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A Comparison Study of Non-Standard Analysis and Non-Archimedean Ultrametric Theory

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Abstract
Infinitesimal elements are mainly considered in non-standard analysis. Recently a new scale free analysis has been developed using the concept of relative infinitesimals, scale free infinitesimals and their corresponding non-archimedean absolute value. With this valuation the real number system R has been extended to an infinite dimensional non-archimedean system R accommodating infinitely small and large numbers. In this paper we present a comparison study of non-standard analysis and non-archimedean ultrametric theory.

Key words: *Hyper Real, Non-archimedean absolute value, Relative Infinitesimal, Scale Invariance.*

1. Introduction

Non-Standard analysis is a branch of classical analysis that formulates analysis using a rigorous notion of an infinitesimal number. Non-Standard analysis was introduced in the early of 1960s by the mathematician Abraham Robinson. Much of the earliest development of infinitesimal calculus by Newton and Leibnitz was formulated using expressions such as infinitesimal number and vanishing quantity. These formulations were widely criticized by George Berkley and others. It was a challenge to develop a consistent theory of analysis using infinitesimals and the first person to do this in a satisfactory way was Abraham Robinson [1]. In 1958 Schmieden and Laugwitz proposed a construction of a ring containing infinitesimals [2]. The ring was constructed from sequences of real numbers. Two sequences were considered equivalent if they differed only in a finite number of elements. Arithmetic operations were defined element wise. However, the ring constructed in this way contains zero divisors and thus cannot be a field.

Since the last few years we have been developing a *Scale Invariant Analysis* [3,4,5] on the set of real numbers R using the concept of relative infinitesimals, scale free infinitesimals and their corresponding absolute value and applying this formalism we extend R into an infinite dimensional metric space R which is a field extension of the set of rational numbers under the new non-Archimedean absolute value [4].

2.1 Approach to Non-Standard Analysis

There are two different approaches to non-standard analysis: the semantic or model theoretic approach and the syntactic approach. Both of these approaches apply to other areas of mathematics beyond analysis, including number theory, algebra, topology and etc.

Robinson's original formulation of non-standard analysis falls into the category of semantic approach. As developed by him in his papers, it is based on studying models (in particular saturated models) of a theory. Since Robinson's work first appeared, a simpler semantic approach (due to

Elias Zakon) has been developed using purely set-theoretic object called super structures. In this approach a model of a theory is replaced by an object called a super structure V(S) over a set S. Starting from a super structure V(S) one construct another object *V(S) using the ultra power construction together with a mapping V(S)→*V(S) which satisfies the transfer principles. The map '*' relates formal properties of V(S) and *V(S). Moreover it is possible to consider a simplified form of saturation called countable saturation. This simplified approach is also more suitable for use by mathematicians who are not specialist in model theory or logic.

Let us briefly recall the ultra power construction of Robinson. Though less direct than the axiomatic approach, it allows one to get a more intuitive contact with the origin of the new structure. Indeed the new infinite and infinitesimal numbers are formulated as equivalence classes of sequences of real numbers, in a way quite similar to the construction of the set of real numbers R from rationals.

Let N be the set of natural numbers. A non-principal (free) ultra filter U on N is defined as follows:

U is a non empty set of subsets of N [P(N) ⊃ U ⊃ φ], such that

I. φ ∈ U
II. A ∈U and B ∈ U ⇒ A ∩ B ∈ U
III. A ∈ U and B ∈ P(N) and B ⊃ A⇒ B ∈ U
IV. B ∈ P(N) ⇒ either B ∈U or { j ∈ N : j ∉ U}∈ U, but not both.
V. B ∈P(N) and B is finite ⇒ B∉ U .

Then the set *R is defined as the set of equivalence classes of all sequences of real numbers modulo the equivalence relation: $a \equiv b$, provided {j : $a_j = b_j$} ∈ U, a and b being two sequences $\{a_j\}$ and $\{b_j\}$.

Similarly, a given relation is said to hold between elements of *R if it holds term wise for a set of indices which belongs to the ultra filter. For example : $a < b$ ⇒ { j : $a_j < b_j$ } ∈ U.

R is isomorphic to a subset of *R, since one can identify any real r ∈ R with the class of sequences $\{r\ ,r, \ldots\ldots\}$. Moreover *R is an ordered field.

The syntactic approach requires much less logic and model theory to understand and use. This approach was developed in the mid 1970s by the mathematician Edward Nelson. Nelson introduced an entirely axiomatic formulation of non-standard analysis that he called Internal Set Theory (IST) [6]. IST is an extension of Zermelo Fraenkel Set Theory (ZST). Along with the basic binary membership relation, it introduces a new predicate standard which can be applied to elements of the mathematical universe together with some axioms of reasoning with this new predicate.

Syntactic non-standard analysis requires a great deal of care in applying the principle of set formation which mathematicians usually take for granted. As Nelson pointed out, a common fallacy in reasoning in IST is that of illegal set formation. For instance, there is no set in IST whose elements are precisely the standard integers (here standard is understood in the sense of new predicate). To avoid illegal set formation, one must only use predicates of Zermelo-Frankel-Choice (ZFC) to define subsets [6].

2.2. Basic Definitions and Constructions of Extended Number Systems

An infinitesimals is a number that is smaller than every positive real number and is larger than every negative real number, or, equivalently, in absolute value it is smaller than 1/m for all m ∈ N ={1,2,3,…..}. Zero is the only real number that at the same time an infinitesimal, so that the non-zero infinitesimals do not occur in standard analysis. Yet, they can be treated in much the same way as can be for the ordinary numbers. For example, each non- zero infinitesimal Є can be inverted and the result is the number ω = 1/Є. It follows that |ω| > m for all m ∈ N, for which reason ω is called hyper large or infinitely large. Hyper large numbers too do not occur in ordinary analysis, but nevertheless can be treated like ordinary numbers. If for example, ω is positive hyper large, we can compute ω/2, ω-1, ω+1, 2ω etc. The positive hyper large numbers must not be confused with ∞, which should not be regarded a number at all.

If Є is hyper small, if δ is too hyper small but non zero and if ω is positive hyper large, so that –ω is negative hyper large, we write Є ≅ 0, δ ≈ 0, ω ≈ ∞, -ω ≈ -∞ respectively. It would be wrong of course, to deduce from ω ≈ ∞ that the difference between ω and ∞ would be hyper small.

Given any x∈R, x ≠ 0 and any δ ≅ 0, let t = x + δ, then Є < |t| < ω, for all Є ≈ 0 and all ω ≈ ∞. The number t is called finite (or appreciable/moderately small or large) number (as it is not too small or not too large).

Three non overlapping sets of numbers (old or new) can now be formed.

a) The set of all infinitesimals, to which zero belongs.
b) The set of all finite numbers, to which all non zero real numbers belong.
c) The set of all hyper large numbers, containing no ordinary numbers at all.

Together these three sets, constitute the set of all numbers of "Real Non-Standard Analysis". This set, which clearly an extension of R, is indicated by *R and is called the * transform of R. The elements of *R are called hyper real.

If a number is not hyper large it is called finite or limited. Clearly, t ∈ *R is finite iff t = x + Є for some x ∈ R and Є ≅ 0. Given such a t, both x and Є are unique, for x + Є = y + δ, x, y Є R, Є, δ ≅ 0, we have $x = y$ (as $x - y$ ∈ R) and ∈ ≅ δ.

By definition x is called standard part of t, and this is written as x = st(t).

The standard part function st provides an important bridge between the finite numbers of non-standard analysis and ordinary real numbers. Trivially, if t is itself an ordinary real number, then st(t) = t.

The * transform not only can be obtained for R but also for N, Z, Q and in fact for any set X of classical mathematics. Their * transforms are indicated by *N, *Z, *Q, *X respectively.

Selecting all finite numbers from *N and *Z we obtain again N, Z, but this is not true for *Q, simply because *Q (just as *R) contains finite non-standard numbers. But again there is a distinct difference between *Q and *R in this respect; there are finite elements t of *Q that cannot be written as t = x + Є, with x ∈ Q, Є ∈ *Q, Є ≈ 0. For let c be any irrational number, say c = $\sqrt{2}$, and let $\{r_1, r_2, \ldots..\}$ be some Cauchy sequence of rationals converging to c. The sequence $\{r_1 - c, r_2 - c, \ldots\ldots\}$ generates an infinitesimals δ in *R (because this sequence converges to zero). On the other hand $\{r_1, r_2, \ldots..\}$ generates an element r ∈ *Q ⊂ *R and r is finite, but it has no standard part in Q, for otherwise $r = x +$ Є for some x ∈ Q and Є ∈ *Q, Є ≅ 0. But $\{r_1 - c, r_2 - c, \ldots..\}$ also generates the finite number $r - c$ ∈ R, so that $r - c = \delta \cong 0$. It follows that $x - c = \delta -$ Є ≅ 0, hence $x - c$ =0 (as $x - c$ is ordinary real), which would mean that c ∈ Q, a contradiction.

There are various ways to introduce new numbers. One way is done by means of infinite sequences of real numbers. In particular, the elements of *R will be generated by means of infinite sequences of reals and it will be necessary to consider all such sequences. (Recall that the elements of R can be generated by means of rather special infinite sequence of rationals, i.e., the Cauchy sequences). More generally, given any set X the elements of its * transform *X will be generated by means of infinite sequence of elements of X, quotionted by the equivalence class generated by the chosen ultra filter and again all such sequences must be taken into account. For example {1,2,3,…….} generates a hyper large element of *N, and {3/2,5/4,9/8,…….} generates a finite element of *Q, and an infinitesimal, generated by {1/2,1/4,1/8,……}. Different sequences may generate the same elements of *X. In fact, given any x ∈ *X there are many (uncountably many) different sequences which form an equivalence class under the ultra filter which represents the element $x \in$ *X. As a consequence, changing finitely many terms of a generating sequence has no effect on the element generated.

2.3 The Purpose of Non-Standard Analysis

Starting from N, the sets Z, Q and R have been introduced in classical analysis in order to enrich analysis with more tools and to refine existing tools. The introduction of negative numbers, of fractions and of irrational numbers is felt as a strong necessity, and without it mathematics would only be a small portion of what it actually is. The introduction of *N, *Z, *Q and *R, however was not meant at all to enrich mathematical analysis (at least not when it all started), but only to simplify it. In fact, definitions and theorems of classical analysis generally are greatly simplified in the context of non-standard analysis. Non-standard analysis has also been applied later in a more traditional way, namely to introduce new mathematical notions and models. Examples can be found in probability theory, asymptotic analysis, mathematical physics, Economics etc.

As an example of a simpler definition, consider *continuity*. A function f from R to R is continuous at c $\in$ R if statement (i) holds:

$\forall\ \epsilon \in R, \epsilon > 0: \exists\ \delta \in R, \delta > 0 : \forall\ x \in R,$

$|x - c| < \delta : |f(x) - f(c)| < \epsilon$ ………...……..(i)

Now to f there corresponds a unique function $*f$, called the * transform of f, that is a function from *R to *R, such that $*f(x) = f(x)$ if $x \in$ R and (i) is true iff (ii), which is the * transform of (i) is true:

$\forall\ \epsilon \in *R, \epsilon > 0 : \exists\ \delta \in *R, \delta > 0 : \forall\ x \in *R,$

$|x - c| < \delta : |*f(x) - *f(c)| < \epsilon$ …...……..(ii)

Moreover (i) is equivalent to much simpler statement (iii)

$\forall\ \delta \in *R, \delta \cong 0 : *f(c + \delta) - *f(c) \cong$ ……(iii)

An illustration of a simpler proof is that of the *Intermediate Value Theorem*:

If : R → R is continuous in the closed interval $[a, b]$, $a < b$, a and b both finite, and $f(a) < 0, f(b) > 0$, then $f(c) = 0$ for some $c \in [a, b]$.

A non-standard proof of this theorem proceeds as follows :

Let, $m \in$ *N be hyper large. Divide $[a, b]$ into m equal subintervals, each of length $\delta = (b - a)/m$. Then $\delta \approx 0$. Let, n be the smallest element of *N such that $*f(a + n\delta) > 0$, then $*f(a + (n - 1)\delta) \le 0$.

Let c = st$(a + n\delta)$, then by continuity, $*f(a + n\delta) - *f(c) = \epsilon_1$ and $*f(c) - *f(a + (n - 1)\delta) = \epsilon_2$ for certain infinitesimals ϵ_1 and ϵ_2. Hence $-\epsilon_1 < f(c) = *f(c) \le \epsilon_2$. But $f(c) \in$ R, so that $f(c) = 0$.

Terrence Tao, one of the most brilliant contemporary mathematicians, has been advocating strongly the use of non-standard analysis as soft analysis rather than using only the *classical hard* analysis in particular differential equations and various other fields of applications in his blog page ‘ What's New’.

So far we have discussed about Robinson's analysis. Our approach is quite different and independent of conventional Non-Standard analysis. In our work we have introduced an infinite dimensional metric space R which is an extension of R using the concept of non-archimedean absolute value.

3. Non-Archimedean Ultra metric Theory

An absolute value on K is a function |.| : K⟶ R_+ that satisfies the following conditions :

$|x| = 0$ iff $x = 0$,

$|xy| = |x||y|$ for all $x, y \in$ K,

$|x + y| \le |x| + |y|$ for all $x, y \in$ K,

We shall say an absolute value of K is non-archimedean if it satisfies the additional condition:

$|x + y| \le \max(|x|, |y|)$ for all $x, y \in$ K

Otherwise, the absolute value is archimedean.

Example- If we take,

$|x| = 1$, if $x \ne 0$ and $|x| = 0$, if $x = 0$ for any field K, then it is trivially a non-archimedean absolute value.

3.1 Non-Archimedean model

Let, *R be a non-standard extension of the real number system R. Let, **0** denote the set of infinitesimals in R. Then an element of *R, denoted as **x**, is written as **x** = x+τ, x∈R and τ∈**0.** The set **0** and hence *R is linearly ordered that matches with the ordering of R. The set **0** is thus of cardinality c, the continuum. The non-zero elements of **0** are new numbers added to R which are constructed from the ring **S** of sequences of real numbers via a choice of an ultrafilter to remove the zero divisors of **S**. A non-standard infinitesimal is realized as an equivalence class of sequences under the ultrafilter and may be considered extraneous to R. The magnitude of an element **x** of *R is evaluated using the usual Euclidean absolute value $|\boldsymbol{x}|_e$.

We now give a new construction relating infinitesimals to arbitrarily small elements of R in a more intrinsic manner. The words ‘’ arbitrarily small elements’’ are made precise in a limiting sense in relation to a scale.

Given an arbitrarily small positive real variable in the sense that $x \to 0^+$, there exists a rational number δ > 0 and a set I_{in}^+ of positive reals $\tilde{x}(x) = \tilde{x}(x, \lambda)$ satisfying $0 < \tilde{x}(x) < \delta < x$ and the inversion rule $\frac{\tilde{x}}{\delta} = \lambda \frac{\delta}{x}$ ……………………………….(I) where 0 < λ(δ) << 1, is a real constant so that $\tilde{x}$ also satisfies the scale invariant equation $x\frac{d\tilde{x}}{dx} = -\tilde{x}$.

The elements $\tilde{x}$ so defined are called relative infinitesimals relative to the scale δ. A relative infinitesimal is negative if -$\tilde{x}$ is a positive relative infinitesimal. The associated scale invariant infinitesimals corresponding to the relative infinitesimals $\tilde{x}$ is defined by $\tilde{X} = \lim_{\delta\to 0}\frac{\tilde{x}}{\delta}$.

Now because of linear ordering of $\mathbf{0}^+$, the set of positive infinitesimals of *R, that is inherited from R, and the fact that the cardinality of $\mathbf{0}^+$ equals that of R, there is a one-one correspondence between $\mathbf{0}^+$ and (0,δ) ⊂ R, which we can write as $\tau(\tilde{x}) = \tau_0\left(\frac{\tilde{x}}{\delta}\right)$ for an infinitesimal $\tau_0 \in \mathbf{0}^+$ and a relative infinitesimal $0 < \tilde{x} < \delta$, $\delta \to 0^+$. This may be interpreted as by saying that for each arbitrarily small δ > 0, there exists in the non-standard *R an infinitesimal $\tau_0 \in \mathbf{0}^+$ so that the dimensionless equality of the form $\frac{\tau}{\tau_0} = \frac{\tilde{x}}{\delta}$ holds good independent of the scale δ. We, henceforth identify $\mathbf{0}^+$ with the set of relative infinitesimals I_{in}^+ in I_δ^+ = (0,δ) ⊂ R so that $I_{in}^+ \subset I_\delta$. We remark that in this framework, a positive variable x is defined relative to the scale δ by the condition x > δ.

A relative infinitesimal $\tilde{x} \in I_{in} \subset I_\delta$ = (-δ,δ) (≠ 0) is assigned with a new absolute value

$v(\tilde{x}) = |\tilde{x}| = \lim_{\delta\to 0}\log_{\delta^{-1}}\widetilde{x_1}^{-1}$, $\widetilde{x_1} = \frac{|\tilde{x}|_e}{\delta}$……..(II). We also set |0| = 0. It is easy to verify that $v(\tilde{x})$ is a non-archimedean absolute value.

Remark 1. We notice that there exists a non-trivial class of infinitesimals (those satisfying $|\tilde{x}|_{e\le\delta\delta^\delta}$) for which the value $|\tilde{x}|$ assigned to an infinitesimal $\tilde{x}$ a real number, i.e., $|\tilde{x}| \ge \delta$. One of our aims is to point out the non-trivial influence of these infinitesimals in real analysis. This is to be contrasted with the conventional approach. The Euclidean value of an infinitesimal in Robinson’s non-standard analysis is numerically an infinitesimal.

Definition (I) is non-trivial in the sense that in absence of it, the scale δ can be chosen arbitrarily close to an infinitesimal $\tilde{x}$, so letting $\delta \to \tilde{x} \to 0^+$, one obtains $|\tilde{x}| = 0$. Thus, dropping the inversion rule, we reproduce the ordinary real number system R with 0 being the only infinitesimal.

Remark 2. An infinitesimal $\tilde{x} \in I_\delta$ has a countable number of different realizations, each for a specific choice of the scale δ, having valuation $\widetilde{|x|}_\delta$. Indeed, given a decreasing sequence of (primary) scales δ_n so that $\delta_n \to 0$ as n→∞, the limit in the defn. II can instead be evaluated over a sequence of secondary smaller scales of the form δ_n^m, $m \to \infty$ for each fixed n.

This observation allows one to extend that definition slightly which is now restated as

i) Scale free (invariant) infinitesimals $\tilde{X}_\delta = \frac{\tilde{x}_n}{\delta^n}$ satisfying $0 < \tilde{x}_n < \delta^n$ (and $\frac{\tilde{x}_n}{\delta^n} = (\frac{\tilde{x}}{\delta})^n$ as $n \to \infty$) are called (positive) scale-free δ-infinitesimals. By inversion, elements of $|\tilde{X}_\delta^{-1}|_e > 1$ are called scale-free δ-infinities.

ii) A relative (δ) infinitesimal $\tilde{x}$ (≠ 0) ∈ I_δ is assigned with a new (δ dependent) absolute value $v(\tilde{x}) = |\tilde{x}|_\delta = \lim_{n\to\infty} \log_{\delta^{-n}} (\frac{\tilde{x}_n}{\delta^n})^{-1}$. (In this scale free notation, all finite real numbers are mapped to 1).

It is easy to verify that |.| defines a non-archimedean absolute value on **0**.

The set **0** is uncountable and the absolute value satisfies the stronger triangle inequality. Accordingly the set **0** = { 0, ± $\delta\widetilde{X_r}$}, r=0,1,…… may be said to acquire the structure of a cantor set like ultra metric space. The set **0** **i**ndeed is realized as a set of nested circle $S_r = \{\tilde{x}_r / v(\tilde{x}_r) = \alpha_r\}$ in the ultra metric norm, when we order, with out any loss of generality, $\alpha_0 > \alpha_1 > \cdots$. The ordinary 0 of R is replaced by this set of scale free infinitesimals $\bar{\mathbf{0}} = \{0, \cup S_r\}$, $\bar{\mathbf{0}}$ being the equivalence class under the equivalence relation ~, where $x \sim y$ means $v(x) = v(y)$.

From the ultra metric property of **0** we can say that

(i) Every open ball in **0** is closed and vice-versa.

(ii) Every point in a ball is centre of the ball.

(iii) Any two balls in **0** are either disjoint or one is contained in another.

(iv) **0** is the union of at most of a countable family of clopen (both open and closed) balls.

(v) The set **0** equipped with the above absolute value is totally disconnected.

Lemma 1. A closed ball in **0** is both complete and compact.

Proof. The proof follows from the following observations. Given ε > 0, consider a closed interval [a,b] ⊂ **0** (in the usual topology) such that 0 < a < b < ε. The valuation v realizes this closed interval as an ultrametric (sub) space U of **0** which is an union of at most of a countable family of clopen balls.

Now we consider completeness. A sequence $\{x_n\}$ ⊂ U is Cauchy ⇔ $v(x_m - x_n) \to 0$ ⇔ $v(x_{n+1} - x_n) \to 0$ ⟹ there exists N > 0 such that $v(x_{n+1}) = v\,(x_n)$ for all n ≥ N. Now since for a non-zero infinitesimal x_n , the associated valuation is non-zero, it follows that $x_n \to x_N$ ∈ U in the ultra metric in the sense that $v(x_n) = v(x_N)$ as $n \to \infty$. Compactness is a consequence of the fact that any sequence in U has a convergent subsequence. Indeed, a sequence $\{x_n\}$ in U can not be divergent in the given ultra metric since $0 \le v(x_n) \le 1$.

Next we extend this non-archimedean structure of **0** on the whole of R.

Let, $I_\delta(r) = r + I_\delta(0)$, $I_\delta(0) = (-\delta, \delta)$, $\delta > 0$ for a real number r ∈ R. For a finite r ∈ R i.e, when $r \notin I_\delta(0)$, we have $\|r\| = |r|_e = r$. For an $r \in I_\delta(0)$, on the other hand $\|r\| = |r| = \lim_{\delta\to 0} \log_{\delta^{-1}} \frac{\delta}{r} = v(r)$ while for an arbitrarily large r (→ ∞) i.e, when $|r|_e > N$, we define $\|r\| = |r^{-1}|$ which is evaluated with the scale δ ≤ 1/N. We notice that the above absolute value awards the real number system R a novel structure i.e, for an arbitrarily small scale δ, numbers x and $\tilde{x}$ satisfying $x > \delta$ and $\tilde{x} < \delta$ now are represented as $x = \delta\delta^{-|\tilde{x}|}$ and $\tilde{x} = \lambda\delta\delta^{|\tilde{x}|}$ [4]. The ultra metric space { R ,‖.‖} is denoted as **R.** From Lemma 1 it is clear that **R** is locally compact, complete metric space. In our paper [4] we have proved that this model **R** is realized as a completion of the field of the rational numbers under this new non-archimedean absolute value ||.||.

4. Conclusion

In this formalism we have presented a new elementary proof of well known *Prime Number Theorem* [4]. Also we have applied this analysis on a class of differential equations [10]. We report in particular, some simple but non-trivial applications of this non-linear formalism leading to emergence of complex non-linear structures even from a linear differential system. It is also shown that anomalous mean square fluctuations can arise naturally from the ordinary diffusion equation interpreted scale invariantly in the present formalism endowing real numbers with a non-archimedean multiplicative structure [5].

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